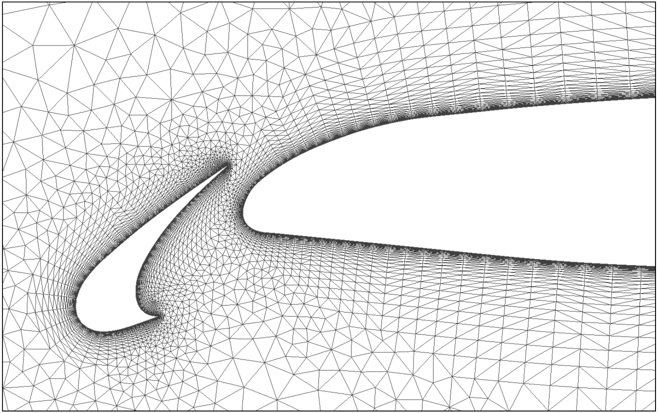
A Robust Spectral PDE Solver for Skinny Triangles

Aaron Yeiser

5/21/2016

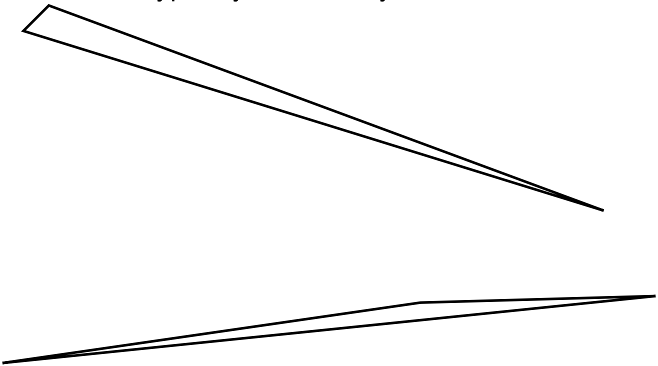
Meshes Can Be Complicated



Fluid flow over an airfoil

Skinny Triangles

When a mesh has skinny triangles, finite element methods are typically numerically unstable.



Two skinny triangles

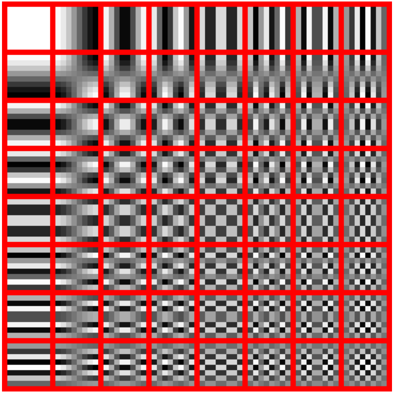
Another Reason for a Robust Method: Computation Time

Remeshing. . . 2%

Remeshing Time $\gg$ Solve Time

Spectral Methods: A Basis

$$f(x) = \sum_i a_i \cdot g_i(x)$$

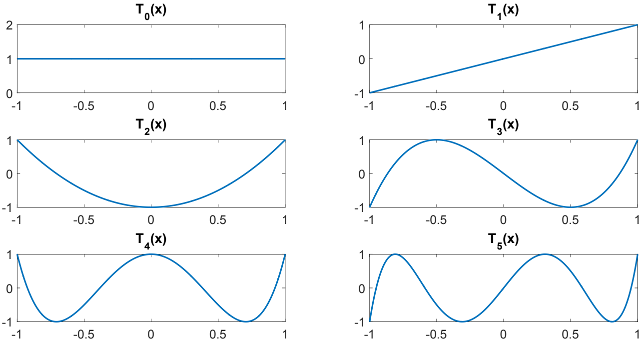


The spectral basis for an 8×8 block of a JPEG image

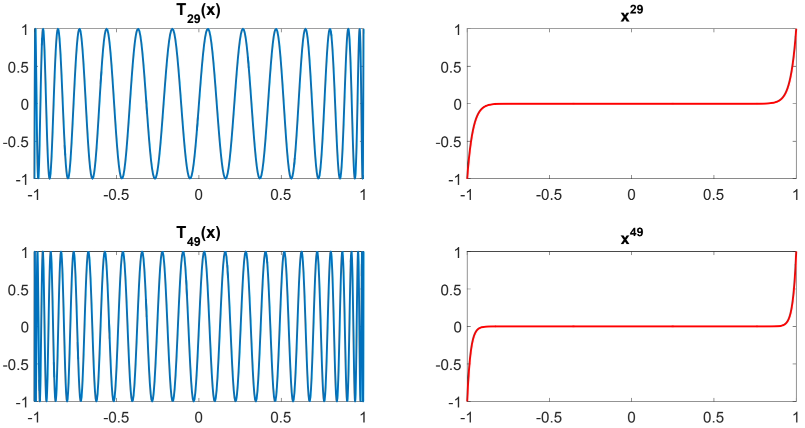
Chebyshev Polynomials: A Good Spectral Basis

$$T_n(x) = \cos(n \arccos(x)), \quad -1 \le x \le 1$$

$$f(x) = \sum_{i=0}^{\infty} a_i T_i(x) \approx \sum_{i=0}^{n} a_i T_i(x)$$



Monomials are a troublesome basis. . .



. . . while 1,000,000-degree Chebyshev expansions have virtually no loss of precision!

Discrete Differential Operators

Differential Equation ⇔ Discrete Linear Operator

$$y(x) = \sum_i y_i T_i(x), \qquad \vec{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix}$$

$$f(x) = \sum_i f_i C_i^{(1)}(x), \qquad \vec{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix}$$

$$D_1 \vec{y} = \vec{f} \quad \Leftrightarrow \quad \frac{dy}{dx} = f$$

$C_n^{(\lambda)}$ are ultraspherical polynomials.

Differential Equations

$$\frac{du}{dx} = f \quad \rightarrow \quad D_1 \vec{u} = \vec{f}$$

$$\frac{du}{dx} + u = f \quad \rightarrow \quad (D_1 + S_0)\, \vec{u} = \vec{f}$$

$$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = f \quad \rightarrow \quad (D_2 - 3S_1 D_1 + 2S_1 S_0)\, \vec{u} = \vec{f}$$

S_k converts a vector from the basis of $C^{(k)}$ to $C^{(k+1)}$ and S_0 converts vectors from a basis of T to a basis of $C^{(1)}$

Boundary Conditions

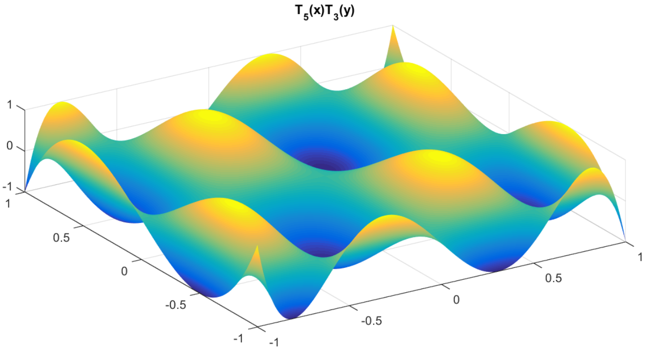
$$\frac{d^2u}{dx^2} = f, \quad u(-1) = lbc, \quad u(1) = rbc$$

$$\begin{bmatrix} 0 & 0 & 4 & & & & \\ & & & 6 & & & \\ & & & & 8 & & \\ & & & & & \ddots & \\ & & & & & & 2n \\ 1 & -1 & 1 & -1 & 1 & \cdots & (-1)^n \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-2} \\ lbc \\ rbc \end{bmatrix}$$

$$T_n(1) = 1, \qquad T_n(-1) = (-1)^n$$

Spectral Methods in Two Dimensions

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} T_i(x) T_j(y)$$



The basis function $T_5(x)T_3(y)$

Two Dimensions are Only a Kronecker Harder than One

$$L \cdot \vec{u} = \vec{f}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

$$u_{xx} \quad + \quad u_{yy} \quad = f$$

$$\left(D_2 \otimes (S_1 S_0) + (S_1 S_0) \otimes D_2\right) \cdot \vec{u} = \vec{f}$$

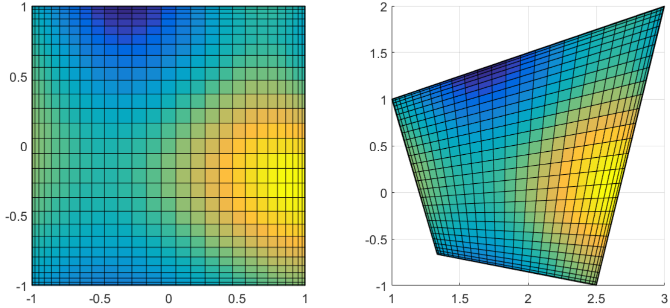
Domains

Problem: Canonical domain is $[-1,1] \times [-1,1]$

Solution: Bilinear Maps!

$$x = a_1 + b_1 x' + c_1 y' + d_1 x' y'$$

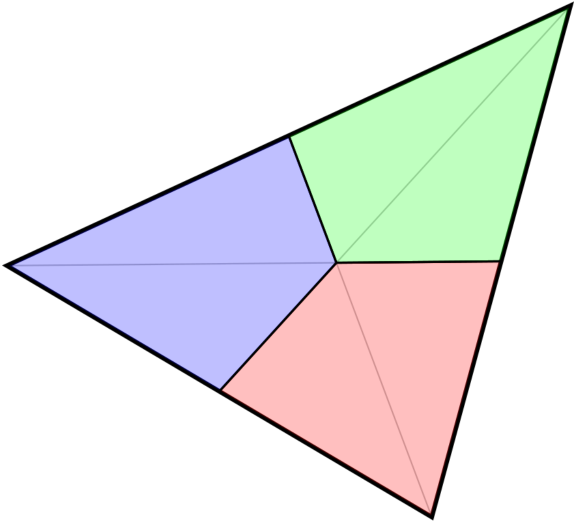
$$y = a_2 + b_2 x' + c_2 y' + d_2 x' y'$$



The Chain Rule is used to transform the differential equations from the quadrilateral to the square.

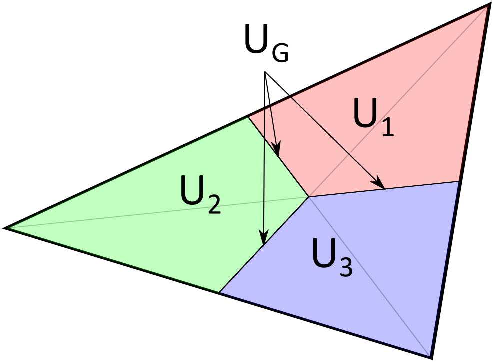
How do we get triangles?

There is no nonsingular transform from a square to a triangle. The solution is to partition the triangle into three quadrilaterals.

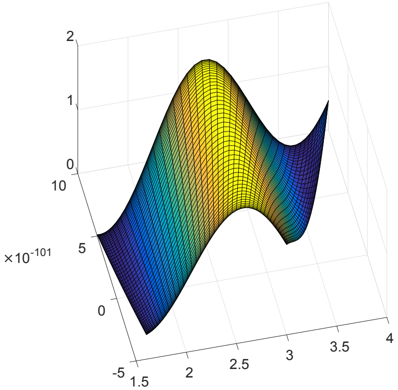


Schur Complement Matrix

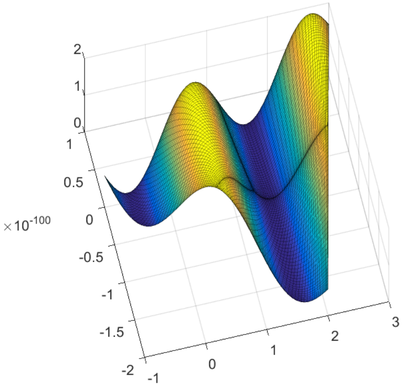
$$\begin{bmatrix} A_{11} & 0 & 0 & A_{1G} \\ 0 & A_{22} & 0 & A_{2G} \\ 0 & 0 & A_{33} & A_{3G} \\ A_{G1} & A_{G2} & A_{G3} & A_{GG} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_G \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_G \end{bmatrix}$$

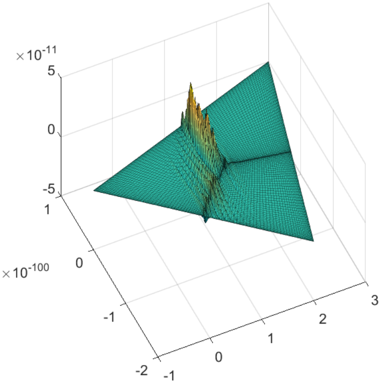


Robustness



Schur Domain Decomposition





Acknowledgements

- Dr. Alex Townsend
- Dr. Khovanova, Dr. Etingof, Dr. Gerovitch, and everyone else involved in organizing the PRIMES-USA program
- Dr. Davis and Dr. Monks (ARML coaches)
- My Parents